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Singular Continuous Spectrum on a Cantor Set of Zero Lebesgue Measure for the Fibonacci Hamiltonian

András Sütő[1,2]

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It is rigorously proven that the spectrum of the tight-binding Fibonacci Hamiltonian, $H_{mn} = \delta_{m,n+1} + \delta_{m,n-1} + \delta_{mn}\mu([(n+1)\alpha] - [n\alpha])$ where $\alpha = (\sqrt{5}-1)/2$ and $[\cdot]$ means integer part, is a Cantor set of zero Lebesgue measure for all real nonzero μ, and the spectral measures are purely singular continuous. This follows from a recent result by Kotani, coupled with the vanishing of the Lyapunov exponent in the spectrum.

KEY WORDS: Schrödinger equation; Cantor spectrum; singular continuity; Lyapunov exponent.

In the present paper we settle a long-standing problem about the spectrum of a one-dimensional discrete Schrödinger operator, called the Fibonacci Hamiltonian. The operator is defined on $l_2(\mathbb{Z})$ by

$$(H\psi)(n) = \psi(n+1) + \psi(n-1) + \mu v(n)\,\psi(n) \tag{1}$$

where $\mu \neq 0$ is a real number,

$$v(n) = [(n+1)\alpha] - [n\alpha] \tag{2}$$

with $\alpha = (\sqrt{5}-1)/2$ and $[\cdot]$ meaning integer part. The spectral problem of H was addressed for the first time independently by Kohmoto *et al.*[1] and by Ostlund *et al.*[2] Nonrigorous but physically appealing arguments, supported by numerical work, led these authors to suggest that the spectrum of H is a Cantor set of zero Lebesgue measure. In a subsequent work,

[1] Université de Lausanne, Institut de Physique Théorique, CH-1015 Lausanne, Switzerland.
[2] On leave from the Central Research Institute for Physics, Budapest, Hungary.

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Kohmoto and Oono[3] found further arguments in favor of a Cantor spectrum, and they also conjectured that the spectral measures should be purely singular continuous for any nonvanishing potential strength μ. Since then, these conjectures about the Cantor spectrum and singular continuity have been widely accepted and propagated in the physical literature as facts, although no rigorous proof has been given. Apparently the first rigorous result was obtained by Casdagli.[4] The main tool of refs. 1–3 was to pass from the Schrödinger equation $H\psi = E\psi$ to the study of a dynamical map $(\tau_{n-1}, \tau_n, \tau_{n+1}) \mapsto (\tau_n, \tau_{n+1}, \tau_{n+2})$ in $\mathbb{R}^3$. Here τ_n is the trace of the transfer matrix M_n defined by

$$\begin{pmatrix} \psi(F_n+1) \\ \psi(F_n) \end{pmatrix} = M_n \begin{pmatrix} \psi(1) \\ \psi(0) \end{pmatrix} \tag{3}$$

where F_n is the nth Fibonacci number. Casdagli proved that

$$B = \{E \in \mathbb{R} \mid \{\tau_n(E)\} \text{ is a bounded sequence}\} \tag{4}$$

is a Cantor set of zero Lebesgue measure if $|\mu| \geqslant 16$. The threshold was improved to $|\mu| \geqslant 4$ by Sütő[5] (without obtaining that the Lebesgue measure $|B| = 0$), who also showed that B coincides with the spectrum of H and that the spectrum is purely continuous for all μ. As a combination of the findings in refs. 4 and 5, one then gets: The spectrum of H is Cantor set of zero Lebesgue measure and it is purely singular continuous if $|\mu| \geqslant 16$. A new development was reached in a recent work by Kotani,[6] who studies operators of the type (1) (Jacobi matrices) with potentials taking values from a finite set. Let $0 \leqslant \theta < 1$, and define an operator $H(\theta)$ of the type (1), but $v(n)$ replaced by

$$v_\theta(n) = [(n+1)\alpha + \theta] - [n\alpha + \theta] \tag{5}$$

Thus, $H(0) = H$. Then, as a consequence of Kotani's result, one has the following.

Theorem (Kotani). For almost all θ, $H(\theta)$ has no absolutely continuous spectrum.

This theorem still cannot exclude that for a given fixed θ, e.g., for $\theta = 0$, there is an absolutely continuous part in the spectrum. In fact, the theorem leaves the possibility for an exceptional set of θ of zero Lebesgue measure, to which 0 may belong. That this is not the case will be proved.

Theorem. For any $\mu \neq 0$ and any θ, the spectrum of $H(\theta)$ is a Cantor set of zero Lebesgue measure. For all θ, $H(\theta)$ has no absolutely continuous spectrum and the spectrum of $H = H(0)$ is purely singular continuous.

Remarks.

(i) The absence of absolutely continuous spectrum for $H(\theta)$ and the singular continuity of the spectral measures of $H(0)$ follow simply from $|\mathrm{spec}(H(\theta))| = 0$ and the continuity of the spectrum of $H(0)$, proved in ref. 5. Indeed, a set of zero Lebesgue measure cannot carry any measure which is absolutely continuous with respect to the Lebesgue one. Notice, however, that for $\theta \neq 0$ the spectrum of $H(\theta)$ may not be purely continuous and thus $H(\theta)$ may have spectral measures composed of singular continuous and pure point parts. Isolated eigenvalues are, however, excluded.

(ii) Claiming something about the set $\mathrm{spec}(H(\theta))$ for all θ is a pleonasm, for $\mathrm{spec}(H(\theta))$ is independent of θ, so is equal to $\mathrm{spec}(H)$. Even more, the θ independence of the integrated density of states (IDS, see ref. 7) holds true,

$$k_\theta(E) = k(E) \tag{6}$$

where $k(E)$ is the IDS of H.

In the proof of the above theorem, we need the equality (6), which is a result by Avron and Simon,[7] valid for almost periodic potentials. As a matter of fact, $\{v(n)\}_{n=-\infty}^{\infty}$ is not a uniformly almost periodic sequence, since $\sup_n |v(n+m) - v(n)| = 1$ for all $m \neq 0$, but it is almost periodic in a generalized sense (for definitions, see Besicovitch[8]): It admits a Fourier series of the form

$$v(n) \sim \sum_{k=-\infty}^{\infty} a_k \exp\{i\lambda_k n\} \tag{7}$$

with $\sum |a_k|^2 < \infty$. Indeed, define

$$f_\alpha(x) = \alpha + \sum_{k=1}^{\infty} \frac{1}{\pi k} \{\sin 2\pi k\alpha(x+1) - \sin 2\pi k\alpha x\} \tag{8}$$

The series $f_\alpha(x)$ is convergent for any real α and x, and

$$f_\alpha(x) = [(x+1)\alpha] - [x\alpha] \tag{9}$$

provided that neither $x\alpha$ nor $x\alpha + \alpha$ is an integer. Therefore, if α is irrational, then

$$[(n+1)\alpha] - [n\alpha] = f_\alpha(n) \tag{10}$$

unless $n=0$ and $n=-1$. The almost periodicity of the sequence (2) can be expressed in another way: One sees easily that for any real θ, v_θ is the pointwise limit of translates of v, and, vice versa, v is the pointwise limit of translates of v_θ. The set of sequences $\{v_\theta \mid 0 \leqslant \theta < 1\}$ is called the hull of v; according to what we said above, it can be obtained from any of its elements by translations and taking limits of translates. This is the property leading to Eq. (6). We remark that Kotani's theorem and the above properties of almost periodicity are valid for any irrational α. In the present paper, we prove our theorem only for $\alpha = (\sqrt{5}-1)/2$. The forthcoming discussion is limited to this case. Generalization will be given elsewhere.

Apart from the result of Kotani, a new ingredient comes from the knowledge of the Lyapunov exponent. The Lyapunov exponent is defined as

$$\gamma(E, \theta) = \lim_{|n| \to \infty} \frac{1}{|n|} \ln \max\{|\psi(n)| : |\psi(0)|^2 + |\psi(1)|^2 = 1\} \tag{11}$$

provided that the limit exists; if this is the case, then

$$\gamma(E, \theta) = \lim_{n \to \infty} \frac{1}{n} \ln \|T_\theta(n)\| \tag{12}$$

where $T_\theta(n)$ is the transfer matrix for the Schrödinger equation $H(\theta)\psi = E\psi$ given by

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = T_\theta(n) \begin{pmatrix} \psi(1) \\ \psi(0) \end{pmatrix} \tag{13}$$

We will write T instead of T_0.

Proposition. For $E \in \mathrm{spec}(H)$, $\gamma(E, 0)$ exists and is equal to zero.

Proof. Due to the symmetry of the potential, $v(-n) = v(n-1)$ for $n \geqslant 2$, it suffices to investigate the limit $n \to \infty$. As a first step, we prove: For $E \in \mathrm{spec}(H)$, $\gamma_n = (1/F_n) \ln \|M_n\|$ tends to zero as $n \to \infty$. From the recurrence equation[1]

$$M_{n+1} = M_{n-1} M_n \tag{14}$$

using $\det M_n = 1$, we obtain

$$M_{n+1} = \tau_n M_{n-1} - M_{n-2}^{-1} \tag{15}$$

It is convenient to work with a norm such that $\|A^{-1}\| = \|A\|$ if $\det A = 1$ (take, e.g., the spectral norm). Then

$$1 \leqslant \|M_{n+1}\| \leqslant |\tau_n| \, \|M_{n-1}\| + \|M_{n-2}\| \tag{16}$$

As it was shown in ref. 5, $\operatorname{spec}(H)=B$, and

$$|\tau_n(E)|<2+|\mu| \tag{17}$$

whenever $\{\tau_n(E)\}$ is a bounded sequence. Thus, there exists some $c_0<\infty$ independent of E and of n, such that

$$1\leqslant \|M_n(E)\|<c_0^n$$

if $E\in\operatorname{spec}(H)$. Finally, we get

$$0\leqslant\gamma_n\leqslant\frac{n}{F_n}\log c_0\to 0 \qquad \text{as} \quad n\to\infty \tag{18}$$

Let now x be any positive integer. There is a unique set of integers $\{n_k\}_{k=0}^K$ such that $n_{k+1}-n_k\geqslant 2$ and

$$F_{n_K}\leqslant x=\sum_{k=0}^{K}F_{n_k}=F_{n_K}+l(x)<F_{n_K+1}$$

Choose $x\geqslant 2$, $x\neq F_n$ for any n. Then[5]

$$v(F_{n_K}+l(x))=v(l(x)) \tag{19}$$

and

$$T(x)=T(l(x))\,T(F_{n_K})=\cdots=T(F_{n_0})\,T(F_{n_1})\cdots T(F_{n_K}) \tag{20}$$

Noticing that $T(F_n)=M_n$, we obtain

$$0\leqslant\ln\|T(x)\|\leqslant\sum_{k\geqslant 0}F_{n_k}\gamma_{n_k} \tag{21}$$

If $E\in\operatorname{spec}(H)$, we can use the upper bound (18) to get

$$0\leqslant\ln\|T(x)\|\leqslant(\log c_0)\sum_{k=0}^{K}n_k\leqslant c_1(\ln x)^2 \tag{22}$$

which proves the assertion. ▮

Observe that (11) and (22) imply that for $E\in\operatorname{spec}(H)$,

$$|\psi(x)|\leqslant c_2\,|x|^{c_1\ln|x|} \qquad \text{as} \quad |x|\to\infty \tag{23}$$

It is expected that (23) can be replaced by some polynomial bound. If $\theta\neq 0$, a recurrence relation like (14) does not exist and we cannot prove that $\gamma(E,\theta)=0$ in the spectrum.

Below we summarize several general results concerning the Lyapunov exponent and valid for the present system.

1. For fixed E, $\gamma(E, \theta)$ exists for a.e. θ and is independent of θ.[9] This quantity is denoted by $\gamma(E)$.

2. There is a formula[10,11] which relates the Lyapunov exponent to the IDS. With $\gamma(E)$ defined above, this reads

$$\gamma(E) = \int \ln |E - E'| \, dk(E') \tag{24}$$

The rigorous proof is due to Avron and Simon[7] and Craig and Simon.[12] Craig and Simon proved that this equation is valid for all E. In the earlier work,[7] where (24) was derived only for a.e. E, Avron and Simon show the following result (Lemma 4.3):

Let

$$\gamma(n; E, \theta) = \frac{1}{n} \ln \|T_\theta(n)\| \tag{25}$$

Then, for any fixed θ, $\gamma(n; E, \theta)$ tends to $\int \ln |E - E'| \, dk(E')$ in $L_2(dE)$.

With (24) this gives, for all θ,

$$\int |\gamma(n; E; \theta) - \gamma(E)|^2 \, dE \to 0 \qquad \text{as} \quad n \to \infty \tag{26}$$

3. Kotani[13] (for the Schrödinger equation on $\mathbb{R}$) and Simon[14] (for the discrete case) obtained the following result.

If $\gamma(E) = 0$ on a subset A of $\mathbb{R}$ with positive Lebesgue measure, then for a.e. θ, the absolutely continuous spectrum of $H(\theta)$ is nonempty in A.

Proof of the Theorem. It is sufficient to show that the spectrum of H has zero Lebesgue measure. This and the continuity of the spectrum of H imply that spec(H) is a Cantor set.

From the Proposition,

$$\gamma(n; E, 0) \to 0 \qquad \text{if} \quad E \in \text{spec}(H)$$

From Eq. (26) we find that $\gamma(E) = 0$ on $A = \text{spec}(H) \backslash S$, where S is a set of zero Lebesgue measure. Thus, if $|\text{spec}(H)| > 0$, then $|A| > 0$, and by Kotani–Simon, $H(\theta)$ has a nonvanishing absolutely continuous spectrum for a.e. θ. This contradicts the theorem of Kotani, so that $|\text{spec}(H)| = 0$. ∎

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NOTE ADDED IN PROOF

The results of ref. 5 and of the present paper have recently been extended to potentials (2) with any irrational α.[15]

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